Journal of Applied Mechanics and Technical Physics, Vol. 45, No. 2, pp. 249–259, 2004

CONTINUOUS RESTRICTED RADIAL MOTION OF A GAS UNDER THE ACTION OF A PISTON

S. V. Khabirov

UDC 517.944+533

The possibility of continuous conjugation of the straight-line radial motion of a gas sphere toward the center and away from the center with the motion where the gas in the entire sphere stops simultaneously is shown. The motion is described by an invariant submodel of rank 1. Time reflections allow one to construct a solution that describes a periodic continuous restricted motion of the gas sphere under the action of a piston.

Key words: *gas sphere, spherical piston, invariant submodel.*

1. Formulation of the Problem. The radial motions of the gas are described by the equations

$$
\begin{gathered}
U_t + UU_r + \rho^{-1} p_r = 0, \\
\rho_t + U\rho_r + \rho(U_r + 2r^{-1}U) = 0, \\
S_t + US_r = 0, \qquad p = f(\rho, S), \qquad a^2 = f_\rho,
\end{gathered}
\tag{1}
$$

where U is the radial velocity, $\rho \geqslant 0$ is the density, p is the pressure, S is the entropy, a is the velocity of sound, and r is the radius of the gas sphere.

The equations admit time transposition ∂_t, uniform extension $t\partial_t + r\partial_r$, and reflection $t' = -t$, $U' = -U$. The invariant solutions were previously considered in [1, § 21].

The regular, partially invariant solution on the subgroup of extensions has the representation

$$U = U(s), \qquad S = S(s), \qquad \rho = \rho(t, r), \qquad s = rt^{-1},$$

whose substitution into system (1) yields the equalities

$$(U - s)S' = 0, \qquad \rho(U - s)U' + tf_\rho \rho_r + f_S S' = 0,$$

$$\rho_t + U\rho_r = -\rho t^{-1}(U' + 2s^{-1}U).$$

The compatibility study yields three cases.

Case 1. $U = s$ and $\rho = \rho_0 t^{-3}\mu(S)$; $\rho = g(p)\mu(S)$ is the equation of state with separated density; $S = S(s)$ is an arbitrary function.

Case 2. $S = S_0$ is a constant; then, we have the integral

$$\frac{U^2}{2} - sU + \int U\, ds + \int f_\rho \rho^{-1}\, d\rho = D(t)$$

and the equation

$$-tD' + (U - s)^2 U' = f_\rho(U' + 2s^{-1}U).$$

The relations are compatible only if $p = a^2\rho + p_0$ (velocity of sound a is constant) and $D = a^2(k \ln t + \ln \rho_0)$ (k is a constant).

Institute of Mechanics, Ural Division, Russian Academy of Sciences, Ufa 450000. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 45, No. 2, pp. 124–135, March–April, 2004. Original article submitted May 5, 2003; revision submitted October 22, 2003.

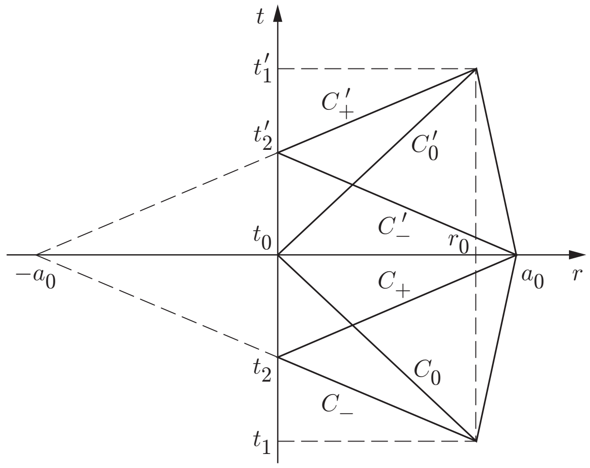

Fig. 1.

Hence, we obtain

$$\rho = \rho_0 t^k \exp\Big[-\frac{u^2}{2} - \int u\,dx\Big], \qquad U = s + au(x), \qquad s = ax.$$

The submodel reduces to the Abel equation of the second kind

$$\frac{dx}{du} = \frac{x(u^2-1)}{2u - x(u^2-k-3)}.$$

Case 3. $S = S_0$, $U = s$, and $\rho = \rho_0 t^{-3}$ for an arbitrary equation of state.

We consider an isentropic radial motion for case 3 with the time transposition:

$$U = r(t-t_0)^{-1}, \qquad \rho = \rho_0|t-t_0|^{-3}, \qquad S = S_0,$$
$$p = f(\rho_0|t-t_0|^{-3}, S_0). \tag{2}$$

The density and pressure increase unrestrictedly as $t \to t_0$.

For a special polytropic gas, we determine the pressure and velocity of sound

$$p = B\rho^{5/3} = p_0|t-t_0|^{-5}, \qquad a = a_0|t-t_0|^{-1}.$$

Here $p_0 = B\rho_0^{5/3}$ and $a_0 = \sqrt{5B/3}\rho_0^{1/3}$. The work of pressure forces on gas compression into a point equals infinity.

The characteristics of system (1) on solution (2), emanating from the point (t_1, r_0), are the straight lines

$$C_0\colon\quad r = r_0\,\frac{t-t_0}{t_1-t_0}, \qquad C_\pm\colon\quad r = r_0\,\frac{t-t_0}{t_1-t_0} \pm a_0\,\frac{t-t_1}{|t_1-t_0|}.$$

Here, t lies between t_0 and t_1. The location of characteristics is shown in Fig. 1. The pattern is invariant on reflection from the straight line $t = t_0$.

For $t_0 > t > t_1$, the characteristic C_- reaches the center ($r = 0$) at the time $t_2 = (r_0t_0 + a_0t_1)(r_0 + a_0)^{-1}$. At $t_0 < t < t_1'$, the characteristic C_+' emanates from the center ($r = 0$) at the time $t_2' = (r_0t_0 + a_0t_1')(r_0 + a_0)^{-1}$ to reach the point (t_1', r_0).

Let the piston compress a special polytropic gas, moving the latter by the law $U = r(t-t_0)^{-1}$, $a = a_0|t-t_0|^{-1}$; at the time t_1, the radius of the gas sphere is r_0, and the pressure approaches the limiting admissible value: $p_1 = p_0|t_1 - t_0|^{-5}$.

We find the motion of the piston $r = R(t)$ and the pressure variation on the piston, such that the gas motion in the sphere stops ($U = 0$) at the time $t = t_2$ when the characteristic C_- arrives at the center ($r = 0$). For this purpose, we solve the boundary-value problem for the system of equations [$a^2 = 5p/(3\rho)$ and $S = S_0$]

$$U_t + UU_r + 3aa_r = 0,$$
$$a_t + Ua_r + 3^{-1}a(U_r + 2r^{-1}U) = 0 \tag{3}$$

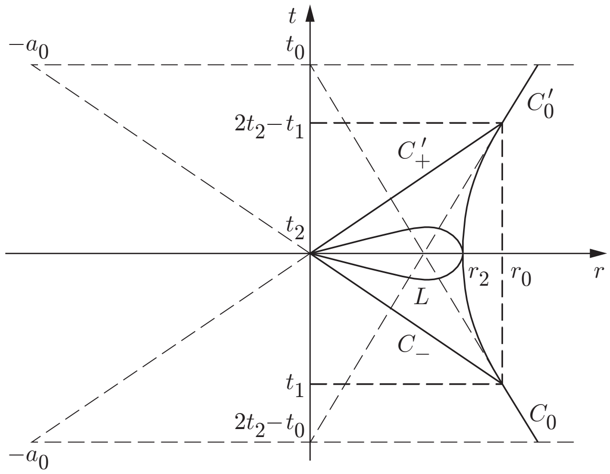

Fig. 2.

in an angular region (Fig. 2) bounded by the straight-line characteristic C_-,

$$r = \frac{a_0(t-t_2)}{t_2-t_0}, \qquad U = \frac{a_0(t-t_2)}{(t_2-t_0)(t-t_0)} < 0, \qquad a = \frac{a_0}{|t-t_0|}, \tag{4}$$

and the stagnation straight line l:

$$t = t_2 = \frac{r_0 t_0 + a_0 t_1}{r_0 + a_0}, \qquad U = 0. \tag{5}$$

The compatibility condition is satisfied at the point $(t_2, 0)$.

The following conditions should be satisfied on the piston $r = R(t)$:

$$R' = U(t, R), \quad R(t_1) = r_0, \qquad p = P(t) \leqslant p_0|t-t_0|^{-5}. \tag{6}$$

From Eqs. (6), we find the law of piston motion by solving the boundary-value problem.

Reflection from the straight line $t = t_2$ yields a solution of the form (2), where the parameter t_0 is replaced by the parameter $2t_2 - t_0$. We obtain the Goursat problem in a spatially similar domain with data on the characteristics C_- and C_+' (see Fig. 2). The problem is invariant with respect to the reflection from the straight line $t = t_2$; therefore, the world lines of particles are symmetric with respect to the straight line $t = t_2$, and the data of the Goursat problem are consistent, i.e., the world line intersects the characteristics at symmetric points with velocities identical in magnitude but opposite in direction. As in [2], we assume that there is a unique analytical solution of the Goursat problem for small r_0. The proof of this fact requires additional considerations.

The problem of conjugation of invariant and partially invariant solutions is included into the SUBMODELS program of gas-dynamics equations [3, 4]. Here, we give an example of continuous conjugation via the characteristic of invariant solutions.

2. Symmetry of the Problem. We introduce new functions by the formulas

$$u = U/3 + a, \qquad v = U/3 - a. \tag{7}$$

System (3) acquires the form

$$u_t + (2u+v)u_r = \frac{v^2-u^2}{2r}, \qquad v_t + (2v+u)v_r = \frac{u^2-v^2}{2r}. \tag{8}$$

Here, the operators in the left sides of equations are differentiations along the characteristics. System (8) is invariant with respect to the following reflections: 1) $u \leftrightarrow v$; 2) $t \to -t$, $u \to -u$, and $v \to -v$; 3) $t \to -t$ and $r \to -r$.

The boundary conditions (4) and (5) take the form

$$C_-: \qquad r = \frac{a_0(t-t_2)}{t_2-t_0}, \qquad u + 2v = \frac{a_0}{t_2-t_0}, \qquad v - u = \frac{2a_0}{t-t_0}; \tag{9}$$

$$l: \qquad t = t_2, \qquad u + v = 0. \tag{10}$$

Theorem 1. *System* (8) *admits the Lie algebra* L_4 *of operators with the basis*

$$X_0 = \partial_t, \qquad X_1 = t\,\partial_t + r\,\partial_r, \qquad X_2 = r\,\partial_r + u\,\partial_u + v\,\partial_v,$$

$$X_3 = t^2\,\partial_t + tr\,\partial_r + (r/3 - tu)\partial_u + (r/3 - tv)\partial_v.$$

Proof. It suffices to write the condition of invariance of system (8) with respect to the operator

$$X = \xi^t\,\partial_t + \xi^r\,\partial_r + \xi^u\,\partial_u + \xi^v\,\partial_v$$

for one equation of the system. For the other equation, this condition is obtained by the substitution $u \leftrightarrow v$.

Equations 2 of the invariance conditions have the form (see [5])

$$\xi^r_v = (2u+v)\xi^t_v, \qquad \xi^r_u = (2v+u)\xi^t_u. \tag{11}$$

Equations 1 of the invariance conditions are

$$\xi^u_v = \frac{v^2-u^2}{2r}\,\xi^t_v, \qquad \xi^v_u = \frac{u^2-v^2}{2r}\,\xi^t_u; \tag{12}$$

$$\begin{aligned} 2\xi^u + \xi^v &= (v-u)\,\frac{v^2-u^2}{2r}\,\xi^t_u + \xi^r_t + (2u+v)[\xi^r_r - \xi^t_t - (2u+v)\xi^t_r],\\ 2\xi^v + \xi^u &= (v-u)\,\frac{v^2-u^2}{2r}\,\xi^t_v + \xi^r_t + (2v+u)[\xi^r_r - \xi^t_t - (2v+u)\xi^t_r]. \end{aligned} \tag{13}$$

We have only to write equations 0 as

$$\begin{aligned} &\Big(\frac{v^2-u^2}{2r}\Big)^2(\xi^t_v - \xi^t_u) + \frac{v^2-u^2}{2r}\,(\xi^u_u - \xi^t_t - \xi^u_v - (2u+v)\xi^t_r + r^{-1}\xi^r) + \xi^u_t + (2u+v)\xi^u_r = r^{-1}(v\xi^v - u\xi^u),\\ &r^{-1}(u\xi^u - v\xi^v) = (2v+u)\xi^v_r + \Big(\frac{v^2-u^2}{2r}\Big)^2(\xi^t_u - \xi^t_v) + + \frac{u^2-v^2}{2r}\,(\xi^v_v - \xi^t_t - \xi^v_u - (2v+u)\xi^t_r + r^{-1}\xi^r) + \xi^v_t. \end{aligned} \tag{14}$$

The linear system (11) has the general solution

$$\begin{aligned} \xi^t &= -\frac{a_u + b_v}{(u-v)^2} + 2\,\frac{a-b}{(u-v)^3},\\ \xi^r &= -\frac{(2v+u)a_u + (2u+v)b_v}{(u-v)^2} + 3(u+v)\,\frac{a-b}{(u-v)^3} + \eta(t,r), \end{aligned} \tag{15}$$

where $a(t,r,u)$ and $b(t,r,v)$ are arbitrary functions.

Equations (13) determine the representations for ξ^u and ξ^v:

$$\begin{aligned} 3\xi^u &= (v-u)\,\frac{v^2-u^2}{2r}\,(2\xi^t_u - \xi^t_v) + \xi^r_t + 3u(\xi^r_r - \xi^t_t) - (7u^2+4uv-2v^2)\xi^t_r,\\ 3\xi^v &= (u-v)\,\frac{u^2-v^2}{2r}\,(2\xi^t_v - \xi^t_u) + \xi^r_t + 3v(\xi^r_r - \xi^t_t) - (7v^2+4uv-2u^2)\xi^t_r. \end{aligned} \tag{16}$$

After substitution of the representations for ξ^t, ξ^r, ξ^u, and ξ^v, Eqs. (12) become equations for a and b, in which we have to separate the variables u and v:

$$\begin{aligned} &(u+v)(2r)^{-1}a'''(u-v)^3 - r^{-1}(2u+3v)a''(u-v)^2 - r^{-1}b''(u-v)^3\\ &\quad + r^{-1}[(3u+11v)a' + (7v-3u)b'](u-v) - a''_t(u-v)^2\\ &\quad + (4a'_t + 2b'_t)(u-v) + [(4u+8v)a'_r + 6vb'_r](u-v) - (2u+v)a''_r(u-v)^2\\ &\quad = 18vr^{-1}(a-b) + 6a_t - 6b_t + (4u+14v)(a_r - b_r); \end{aligned} \tag{17}$$

the second equation is obtained by replacing $u \leftrightarrow v$ and $a \leftrightarrow b$ in (15). Here, $a' = a_u$ and $b' = b_v$.

In equality (17), we pass to the limit as $u \to v$ and obtain

$$18r^{-1}va + 6a_t + 18va_r = 18r^{-1}vb + 6b_t + 18vb_r. \tag{18}$$

By virtue of (18), the right side of equality (17) can be presented as

$$18vr^{-1}(a(u)-a(v))+6(a_t(u)-a_t(v))-4(u-v)b_r(v)-18v(a_r(v)-a_r(u))+4(u-v)a_r(u).$$

Dividing Eq. (17) by $(u-v)$ and passing to the limit as $u \to v$, we obtain

$$4vr^{-1}a' + 2a'_t + 6va'_r + 4a_r = 4vr^{-1}b' + 2b'_t + 6vb'_r + 4b_r. \tag{19}$$

By virtue of (18) and (19), equality (17) is presented in the form

$$[(2r)^{-1}(u+v)a'''(u) - r^{-1}b''(v)](u-v) = r^{-1}(2u+3v)a''(u) + a''_t(u) + (2u+v)a''_r(u)$$

$$+ 18vr^{-1}\,\frac{a(u)-a(v)-a'(v)(u-v)}{(u-v)^2} - 3r^{-1}a'(u) - 14vr^{-1}\,\frac{a'(u)-a'(v)}{u-v}$$

$$+ 6\,\frac{a_t(u)-a_t(v)-a'_t(v)(u-v)}{(u-v)^2} - 4\,\frac{a'_t(u)-a'(v)}{u-v} + 18v\,\frac{a_r(u)-a_r(v)-a'_r(v)(u-v)}{(u-v)^2}$$

$$- 12v\,\frac{a'_r(u)-a'_r(v)}{u-v} + 4\,\frac{a_r(u)-a_r(v)}{u-v} - 4a'_r(u) + 3r^{-1}b'(v).$$

Passing to the limit as $u \to v$, we obtain $b' = a'$; then, from (18) and (19), it follow that $a_r = b_r$ and $a_t = b_t$. Hence, we have $a(t,r,v) = b(t,r,v)$ and Eq. (17) becomes a functional equation for determining a. We present the functions $a(u)$, $a_r(u)$, $a_t(u)$, $a'(u)$, … in the form of series in powers of $(u-v)$, and Eq. (17) becomes

$$\frac{(u-v)+2v}{2r}\Big[a'''(v) + \sum_{k=1}\frac{1}{k!}\,a^{(k+3)}(v)(u-v)^k\Big] - \frac{a''(v)}{r}$$

$$= \frac{18v}{r}\sum_{k=3}\frac{1}{k!}\,a^{(k)}(v)(u-v)^{k-3} - \frac{14v}{r}\sum_{k=2}\frac{1}{k!}\,a^{(k+1)}(v)(u-v)^{k-2}$$

$$+ \frac{5v}{r}\sum_{k=1}\frac{1}{k!}a^{(k+2)}(v)(u-v)^{k-1} + \frac{2a''(v)}{r} + \frac{2}{r}\sum_{k=1}\frac{1}{k!}\,a^{(k+2)}(v)(u-v)^k$$

$$- \frac{3}{r}\sum_{k=1}\frac{1}{k!}\,a^{(k+1)}(v)(u-v)^{k-1} + 6\sum_{k=3}\frac{1}{k!}\,a_t^{(k)}(v)(u-v)^{k-3}$$

$$- 4\sum_{k=2}\frac{1}{k!}\,a_t^{(k+1)}(v)(u-v)^{k-2} + \sum_{k=1}\frac{1}{k!}\,a_t^{(k+2)}(v)(u-v)^{k-1}$$

$$+18v\sum_{k=3}\frac{1}{k!}\,a_r^{(k)}(v)(u-v)^{k-3} - 12v\sum_{k=2}\frac{1}{k!}\,a_r^{(k+1)}(v)(u-v)^{k-2}$$

$$+ 3v\sum_{k=1}\frac{1}{k!}\,a_r^{(k+2)}(v)(u-v)^{k-1} + 2a''_r(v) + 2\sum_{k=1}\frac{1}{k!}a_r^{(k+2)}(v)(u-v)^k$$

$$+ 4\sum_{k=2}\frac{1}{k!}\,a_r^{(k)}(v)(u-v)^{k-2} - 4\sum_{k=1}\frac{1}{k!}\,a_r^{(k+1)}(v)(u-v)^{k-1}.$$

Equating the coefficients at the powers of $(u-v)^k$ $(k = 1,2,3)$ to zero, we obtain

$$\begin{gathered} vr^{-1}a^{IV} = a_t^{IV} + 3va_r^{IV} + 8a'''_r, \\ 2vr^{-1}a^{V} = a_t^{V} + 3va_r^{V} + 10a_r^{IV}, \\ a_t^{VI} + 3va_r^{VI} + 12a_r^{V} = 3vr^{-1}a^{VI} + 5(2r)^{-1}a^{V}. \end{gathered} \tag{20}$$

We eliminate the derivatives with respect to the variable t:

$$va^{V} + ra_r^{IV} = a^{IV}, \qquad va^{VI} + ra_r^{V} = -2^{-1}a^{V}.$$

Eliminating the derivatives with respect to r, we obtain $a^V = 0$. Then, it follows from Eq. (20) that $a_r^{IV} = 0$, $vr^{-1}a^{IV} = a_t^{IV} + 8a_r''' \Rightarrow a^{IV} = 0$, $a''' = -6\xi(t) \Rightarrow$

$$a = -\xi u^3 + k_2 u^2 + k_1 u + k_0, \qquad b = -\xi v^3 + k_2 v^2 + k_1 v + k_0,$$

where $\xi(t)$ and $k_i(t, r)$ are arbitrary functions. Substitution into Eqs. (15) and (16) yields $\xi^t = \xi(t)$, $\xi^r = \eta(t, r)$, $\xi^u = \eta_t/3 + u(\eta_r - \xi_t)$, and $\xi^v = \eta_t/3 + v(\eta_r - \xi_t)$.

Equations 0 of system (14) are separated in terms of the variables u and v:

$$r\eta_r = \eta, \qquad \eta_{tt} = 0, \qquad r\xi_{tt} = 2\eta_t \qquad \Longrightarrow$$

$$\xi^r = r(C_1 t + C_0), \qquad \xi^t = C_1 t^2 + C_2 t + C_3,$$

$$\xi^u = (r/3 - tu)C_1 + (C_0 - C_2)u, \qquad \xi^v = (r/3 - tv)C_1 + (C_0 - C_2)v.$$

Theorem 2. *There exists a unique operator Y from the algebra L_4, which retains invariant the boundary condition* (9) *on the characteristic C_-.*

The boundary conditions on the characteristic C_- and straight line l are written via the invariants of the operator Y.

Proof. We write the general operator of the algebra L_4 in the form

$$X = x^0 X_0 + x^1 X_1 + x^2 X_2 + x^3 X_3 = (x^0 + x^1 t + x^3 t^2)\,\partial_t + (x^1 + x^2 + x^3 t) r\,\partial_r$$

$$+ (x^2 u + x^3(r/3 - tu))\,\partial_u + (x^2 v + x^3(r/3 - tv))\,\partial_v.$$

The criterion of invariance of the manifold C_- yields the relations

$$x^2 = t_2 x^3, \qquad x^0 = t_0 t_2 x^3, \qquad x^1 = -(t_0 + t_2)x^3.$$

Hence, the manifold C_- is invariant with respect to the operator

$$Y = t_0 t_2 X_0 - (t_0 + t_2)X_1 + t_2 X_2 + X_3$$

$$= (t - t_0)(t - t_2)\,\partial_t + (t - t_0)r\,\partial_r + (r/3 - (t - t_2)u))\,\partial_u + (r/3 - (t - t_2)v))\,\partial_v.$$

The invariants of the operator Y are defined by the basis

$$r^{-1}(t - t_2) = s < 0, \qquad (t - t_0)(su - 1/3) = u_1(t_0 - t_2),$$

$$(t - t_0)(sv - 1/3) = v_1(t_0 - t_2).$$

The boundary conditions (9) and (10) are written via the invariants

$$C_-\colon \qquad s = a_0^{-1}(t_2 - t_0) = s_0 < 0, \qquad u_1 = 4/3, \qquad v_1 = -2/3; \tag{21}$$

$$l\colon \qquad u_1 + v_1 = 2/3, \qquad s \to -0. \tag{22}$$

The boundary condition (10) is stronger than (22),

$$U = \Big(\frac{1 - 3/2(u_1 + v_1)}{s} - \frac{r}{t_0 - t_2}\Big)\Big(1 - \frac{rs}{t_0 - t_2}\Big)^{-1} \longrightarrow 0 \tag{23}$$

as $s \to -0$. Condition (23) is not invariant with respect to the operator Y. Here, u_1 and v_1 are functions of the variables s and r.

3. Invariant Solutions. The solution of the boundary-value problem (8), (21), (22) can be sought as an invariant Y-solution of the form

$$\begin{aligned} u &= s^{-1}\Big(1/3 + (t_0 - t_2)(t - t_0)^{-1}u_1(s)\Big), \\ v &= s^{-1}\Big(1/3 + (t_0 - t_2)(t - t_0)^{-1}v_1(s)\Big). \end{aligned} \tag{24}$$

Substitution into (8) yields the invariant submodel

$$\begin{aligned} -(2u_1 + v_1)su_1' &= 2^{-1}(v_1 - u_1)^2 - 3u_1^2 + u_1, \\ -(2v_1 + u_1)sv_1' &= 2^{-1}(u_1 - v_1)^2 - 3v_1^2 + v_1. \end{aligned} \tag{25}$$

The invariant solution (24) is conjugate to solution (2) and is determined in the domain $0 < s < s_0$. The following condition is satisfied: on the curve L (see Fig. 2)

$$(2/3)rs + (t_2 - t_0)(2/3 - u_1 - v_1) = 0 \tag{26}$$

the equality $U = 0$ holds.

Submodel (25) reduces to the self-similar equation

$$(2u_1 + v_1)[(v_1 - u_1)^2/2 - 3v_1^2 + v_1]\,du_1 = (2v_1 + u_1)[(v_1 - u_1)^2/2 - 3u_1^2 + u_1]\,dv_1 \tag{27}$$

and the quadrature

$$-\frac{ds}{s} = \frac{(2u_1 + v_1)\,du_1}{(v_1 - u_1)^2/2 - 3u_1^2 + u_1} = \frac{(2v_1 + u_1)\,dv_1}{(v_1 - u_1)^2/2 - 3v_1^2 + v_1}. \tag{28}$$

Equation (27) has been studied in the papers [6; 7; 8, § 162] on self-similar compressions of a quiescent gas sphere under the action of a spherical piston. Here, Eq. (27) is a submodel of the projective operator Y for finding the invariant solution conjugate to the radial solution (2). The invariant submodels are identical for different subalgebras: X_1 solutions of the form $u = s^{-1}(1/3 - u_1)$, $v = s^{-1}(1/3 - v_1)$, $s = (t - t_2)r^{-1}$ and Y solutions of the form (24) are determined by submodel (25). Such invariant solutions are related by the transformation

$$u' = \frac{t_2 - t_0}{t - t_0}\,u + \frac{r}{3(t - t_0)}, \qquad v' = \frac{t_2 - t_0}{t - t_0}\,v + \frac{r}{3(t - t_0)},$$

which is a transformation of similarity only for the invariant solutions considered. System (8) is not invariant with respect to these transformations; therefore, the subalgebras X_1 and Y are not similar.

We describe the results of a qualitative study of the behavior of integral curves. Equation (27) has the integral straight lines l_1: $u_1 = v_1$ and l_2: $u_1 + v_1 = 2/3$. The pattern of integral curves is symmetric with respect to the straight line $u_1 = v_1$, because Eq. (27) is invariant with respect to the substitution $u_1 \leftrightarrow v_1$. There are six singular points.

Point O ($u_1 = v_1 = 0$) is a double node with common tangential lines of the integral curves $u_1 = 0$ and $v_1 = 0$.

Points B_1 ($u_1 = -2/3$, $v_1 = 4/3$) and B ($u_1 = 4/3$, $v_1 = -2/3$) are degenerate nodes with a common tangential line l_2 of the integral curves. The point B coincides with conditions (21) on the characteristic C_-.

Point A ($u_1 = v_1 = 1/3$) is a proper node satisfying condition (22) on the straight line l.

Point C [$u_1 = (1 + \sqrt{3})/6$, $v_1 = (1 - \sqrt{3})/6$] is a saddle with separatrices tangential to the straight lines

$$v_1 - (1 - \sqrt{3})/6 + (\sqrt{3} + 1)(\sqrt{3} \mp \sqrt{5/2})(u_1 - (1 + \sqrt{3})/6) = 0.$$

Point C_1 [$u_1 = (1 - \sqrt{3})/6$, $v_1 = (1 + \sqrt{3})/6$] is a saddle with separatrices tangential to the straight lines

$$v_1 - (1 + \sqrt{3})/6 + (\sqrt{3} - 1)(\sqrt{3} \mp \sqrt{5/2})(u_1 - (1 - \sqrt{3})/6) = 0.$$

The integral curves have directions with a zero slope to the v_1 axis at points of the straight line m_1 ($2v_1 + u_1 = 0$) and the hyperbola g_1 [$6(u_1 - v_1)^2 = (6u_1 - 1)^2 - 1$]; the integral curves have directions with a zero slope to the u_1 axis at points of the straight line m_2 ($2u_1 + v_1 = 0$) and the hyperbola g_2 ($6(u_1 - v_1)^2 = (6v_1 - 1)^2 - 1$) (Fig. 3).

The hyperbolas g_1 and g_2 and the straight lines m_1 and m_2 are lines of changes in sign in expression (28). The arrows in Fig. 3 indicate the direction of motion over the integral curves for the inequality $ds > 0$ to be satisfied. There are two straight lines m_1 and m_2 on which the direction of the arrows changes.

The position of hyperbolas and separatrices in the saddle is determined by their slopes. The position of the hyperbola and the tangential line of the integral curves in the degenerate node are determined in a similar manner.

The boundary condition (22) either determines the integral curve l_2 or separates the integral curves passing through the point A.

Thus, the integral curve connecting the degenerate node B and the proper node A describes the solution of the boundary-value problem (8), (9), (26), if the expression ds does not change its sign along this integral curve. It is seen from Fig. 3 that there are many curves of this kind: $v_1 = \varphi(u_1, k)$, k is a parameter of the curves, such that $v_1 - 1/3 \sim -k(u_1 - 1/3)$ as $u_1 \to 1/3$, $1 < k < k_0 = 2 + \sqrt{3}$. The curves are located between the straight line $u_1 + v_1 = 2/3$ and separatrices of the saddle passing into the degenerate node and proper node: $\varphi(4/3, k) = -2/3$, $\varphi(1/3, k) = 1/3$, $\varphi'(4/3, k) = -1$, and $\varphi'(1/3, k) = -k$.

The integral curve $u_1 + v_1 = 2/3$ corresponds to the solution of problem (25), (21), (22): $u_1 = 1/3 + ss_0^{-1}$ and $v_1 = 1/3 - ss_0^{-1}$. Formulas (7) and (24) give the distribution of the gas parameters (2).

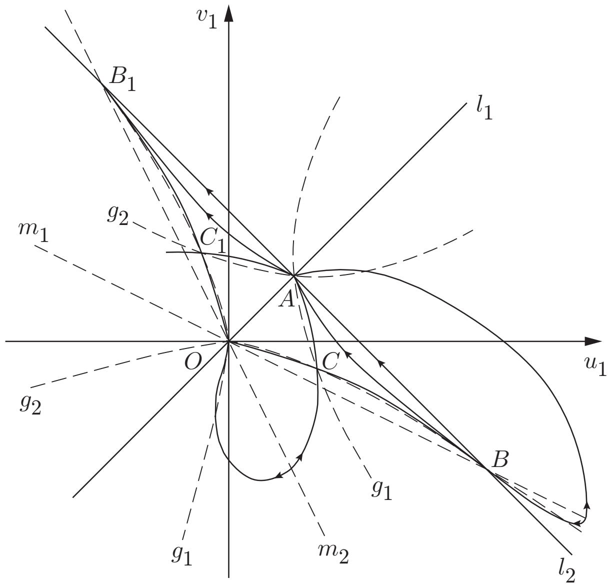

Fig. 3.

Lemma 1. *For all integral curves entering the proper node, we have* $s \to 0$.

Proof. In the linear approximation, the integral curve has the form $v_1 \simeq 1/3 - k(u_1 - 1/3)$ for $u_1 \to 1/3$. Equation (28) is written as

$$-\frac{ds}{s} \simeq \Big(-\frac{3}{3u_1 - 1} + \frac{3(k^2-1)}{3((k+1)^2-6)u_1 - (k+1)^2}\Big)\,du_1.$$

Hence, we have

$$-s_0^{-1}s \simeq (u_1 - 1/3)\Big(1 - ((k+1)^2/2 - 3)(u_1 - 1/3)\Big)^{\frac{1-k^2}{(k+1)^2-6}}$$

as $u_1 \to 1/3$. Consequently, $s \to -0$, $u_1 \simeq 1/3 - s_0^{-1}s$, and $v_1 \simeq ks_0^{-1}s + 1/3$.

The asymptotic behavior of the solution of system (24) in the neighborhood of the point $s = 0$, $u_1 = v_1 = 1/3$ is

$$u_1 = \frac{1}{3} + s_1\Big(-1 + \sum_{j=1}^{\infty} a_j s_1^j\Big), \qquad v_1 = \frac{1}{3} + s_1\Big(k + \sum_{j=1}^{\infty} b_j s_1^j\Big), \tag{29}$$

where

$$s_1 = ss_0^{-1}, \qquad -b_1 = a_1 = (1-k^2)/2, \qquad b_2 = ka_2, \qquad a_2 = a_1/2,$$

$$(j+1)a_{j+1} = -kb_j - (1 + (k-2)j)a_j - \sum_{i=1}^{j-1}\Big[\frac{b_i b_{j-i}}{2} + \Big(2(j-i) - \frac{1}{2}\Big)a_i a_{j-i} + (j-i)a_i b_{j-i}\Big],$$

$$(j+1)b_{j+1} = a_j + (k - (2k-1)j)b_j - \sum_{i=1}^{j-1}\Big[\frac{a_i a_{j-i}}{2} + \Big(2(j-i) - \frac{1}{2}\Big)b_i b_{j-i} + (j-i)a_i b_{j-i}\Big], \qquad j \geqslant 2.$$

On the invariant solution, Eq. (23) yields

$$U \to \frac{3(1-k)}{2s_0} + \frac{r}{a_0 s_0} \qquad \text{for} \quad s \to 0$$

and $U = 0$ for $r = r_2 = 3a_0(k-1)/2$. Hence, we have $U \to 3(1-k)/(2s_0) \neq 0$ as $r \to 0$ and $k \neq 1$ along $t = t_2$. The slope of characteristics increases monotonically with increasing r, which corresponds to compression. Characteristics of one family do not intersect each other in the domain $0 < s < s_0$.

4. Piston Motion. To find the law of piston motion, pressure variation on the piston, and piston position r_2 at $t = t_2$, we have to solve problem (6).

Introducing the variable $s_1 = r_1^{-1}(t - t_2)$ ($s_1 = ss_0^{-1}$ and $r_1 = ra_0^{-1}$) instead of t, we obtain

$$\frac{dr_1}{r_1} = \Big[\frac{2(r_1 s_1 + 1)}{3(u_1 + v_1)} - 1\Big]\frac{ds_1}{s_1}. \tag{30}$$

The initial data $r = r_2 = 3a_0(k - 1)$ determine the motion of the piston orthogonally approaching the straight line $t = t_2$. For $s_1 = 1$, the piston position $r = r_0$ on the characteristic $s = s_0$ of the straight-line radial motion is determined uniquely.

Reflection from the straight line $t = t_2$ yields a continuation of the solution through the velocity discontinuity (velocity changes its sign); the velocity of sound is continuous. The condition of the contact discontinuity is simultaneously satisfied at all points of the sphere $r \leqslant r_2$.

5. Variation of Constants of the Invariant Solution. The invariant solution (24) depends on two constants: k and s_0. We present the solution of system (8) in the form

$$u = \frac{1}{s}\Big(\frac{1}{3} - \frac{u_1(k, n)}{s_1 r_1 + 1}\Big), \qquad v = \frac{1}{s}\Big(\frac{1}{3} - \frac{v_1(k, n)}{s_1 r_1 + 1}\Big), \tag{31}$$

where $s = (t - t_2)r^{-1}$, $r_1 = ra_0^{-1}$, $s_1 = ss_0^{-1}$, $k = k(s_1, r_1)$, and $n = n(s_1, r_1)$; the functions $u_1(k, s_1)$ and $v_1(k, s_1)$ determine the presentation of the invariant solution (29).

The asymptotic behavior $n \sim s_1$, $k \sim 2r_1/3 + 1$ as $s_1 \to 0$ provides orthogonality of the world lines to the straight line $t = t_2$. We require that $U = 0$ for $r = 0 \Longrightarrow u_1 + v_1 = 2/3$ ($r_1 = 0$).

The conditions $n = 1$ and $k_{r_1} = 0$ for $s_1 = 1$ ensure conjugation of the solution to the straight-line radial motion (2).

Substitution of (31) into (8) yields the equations for the functions n and k:

$$\begin{gathered}(2u_1 + v_1)[u_{1k}s_1k_{s_1} + u_{1n}(s_1n_{s_1} - n)] + (1 + s_1r_1 - 2u_1 - v_1)r_1(u_{1k}k_{r_1} + u_{1n}n_{r_1}) = 0,\\ (2v_1 + u_1)[v_{1k}s_1k_{s_1} + v_{1n}(s_1n_{s_1} - n)] + (1 + s_1r_1 - 2v_1 - u_1)r_1(v_{1k}k_{r_1} + v_{1n}n_{r_1}) = 0.\end{gathered} \tag{32}$$

This system becomes degenerate at $s_1 = 0$ and $r_1 = 0$. In the neighborhood of $s_1 = 0$, we construct a solution in the form of a series in powers of s_1, which satisfies the required asymptotic behavior for $s_1 \to 0$ and $r_1 = 0$.

Indeed, if $n = s_1 + n_2s_1^2 + n_3s_1^3 + \ldots$, $k = 1 + 2r_1/3 + k_1s_1 + \ldots$, then it follows from (32) that $n_2 = 0$, $k_1 = 2r_1/3(1 + r_1/3)$, $n_3 = -2r_1/3(1 + r_1/3)(1 + 2r_1/3)$, $\ldots$. By virtue of the expansion $u_1 + v_1 - 2/3 = n(k-1) + O(n^2) \sim 2r_1s_1/3(1 + (1 + r_1/3)s_1 + \ldots) \to 0$ as $r \to 0$, the condition at the boundary $r = 0$ is satisfied.

This is the way the asymptotic behavior of the solution of the Goursat problem is constructed in the neighborhood of the straight line $t = t_2$ for $s < 0$.

6. Reflected Solution. We consider a solution that is a reflection of the constructed solution from the straight line $t = t_2$. It describes continuous restricted expansion of the gas up to the straight-line radial motion by law (2), where t_0 is replaced by $2t_2 - t_0$. The characteristic C_- is transformed to the characteristic

$$C'_+: \qquad r = a_0\frac{t - t_0}{t_0 - t_2}, \qquad 2u + v = \frac{a_0}{t_0 - t_2}, \qquad v - u = -\frac{2a_0}{t - 2t_2 + t_0}. \tag{33}$$

Condition (5) on the straight line l remains unchanged: $u + v = 0$.

Manifold (33) is invariant with respect to the operator similar to the operator Y from Theorem 2, where t_0 is replaced by $2t_2 - t_0$.

The invariants and the form of the invariant solution change accordingly:

$$u = s^{-1}(1/3 + (t_2 - t_0)(t + t_0 - 2t_2)^{-1}u_1(s)),$$

$$v = s^{-1}(1/3 + (t_2 - t_0)(t + t_0 - 2t_2)^{-1}v_1(s)), \qquad s = (t - t_2)r^{-1}.$$

Substitution into system (3) leads to system (25) with the boundary conditions C'_+: $s = (t_0 - t_2)a_0^{-1} = -s_0 > 0$, $u_1 = -2/3$, $v_1 = 4/3$, and l: $u_1 + v_1 = 2/3$ as $s \to 0$. These conditions correspond to the integral curves of Eq. (27) connecting the singular point A (proper node) and the singular point B_1 (degenerate node), along which the value of s ($ds > 0$) monotonically increases if we move along the arrow (see Fig. 3). These integral curves are symmetric with respect to the bisector $u_1 = v_1$ to the integral curves $v_1 = \varphi(u_1, k)$ solving the problem

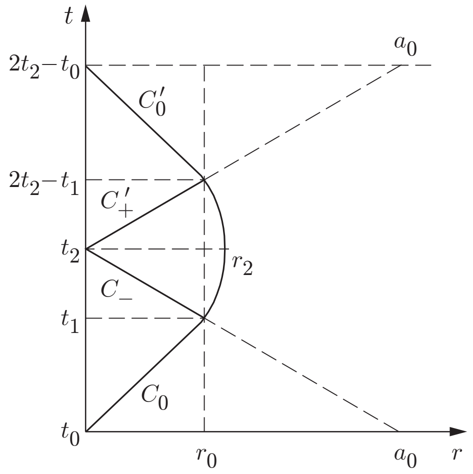

Fig. 4.

of continuous restricted compression of the gas until its stop on curve (26), i.e., they are represented by the formula $u_1 = \varphi(v_1, k)$ and the asymptotic curve $u_1 - 1/3 \sim -k(v_1 - 1/3)$ as $v_1 \to 1/3$.

The boundary condition (23) becomes

$$\frac{2/3 - u_1 - v_1}{s_1} \longrightarrow \frac{2r_1}{3} \qquad \text{for} \quad s_1 \to 0.$$

The equation for the world line coincides with problem (30) where s_0 is replaced by $-s_0$ and is considered for $s > 0$. The problem is invariant with respect to the substitution $s_0 \to -s_0$, $s \to -s$, $u_1 \leftrightarrow v_1$; therefore, the answer is written by the solution of problem (30). Variation of the constants of the reflected invariant solution yields the asymptotic solution for the Goursat problem in the neighborhood of the straight line $t = t_2$ for $s > 0$.

7. Periodic Motion. We consider the problem of a continuous transformation of the gas expanding by law (2) into the gas compressing by a similar law (reflection from a certain straight line $t = \text{const}$).

Let the piston moving along the straight line C_0: $r(t_1 - t_0) = r_0(t - t_0)$ change its motion at the time $t = t_1$ so that the gas in the entire sphere stops at the time $t = t_2$, and the motion is smooth for all t. The time $t_2 = (r_0 t_0 - a_0 t_1)(r_0 - a_0)^{-1}$ is determined by the intersection of the characteristic C_- on solution (2) with the axis t (Fig. 4). The inequality $r_0 < a_0$ should hold; otherwise, the straight-line characteristic C_- would not cross the axis t. If this motion is constructed, then the reflection from the straight line $t = t_2$ yields the solution of the posed problem.

We seek the solution of Eq. (3) in the angular domain bounded by the characteristic C_-: $r = a_0(t - t_2)(t_0 - t_2)^{-1}$, where the functions $U = a_0(t - t_2)(t_0 - t_2)^{-1}(t - t_0)^{-1} > 0$ and $a = a_0(t - t_0)^{-1}$ are specified, and the stagnation straight line l: $t = t_2$, where the velocity is $U = 0$. The no-slip condition (6) is satisfied on the piston $r = R(t)$. From this condition, we find the world lines of particles.

The problem is invariant with respect to the operator Y from Theorem 2; hence, the solution can be found in the form (24) for Eqs. (27) and (28) with conditions that coincide with (21) and (22). Hence, the solution $u_1(s)$, $v_1(s)$ is the same as that in the problem of transforming the compressing gas into the expanding gas.

In calculating the world lines, the difference in formulas (30) is only the sign of the expression $t_2 - t_0 = -a_0 s_0$. Then, all considerations from Secs. 3–6 are valid.

Consecutive conjugation of radial expansion and radial compression leads to a periodic continuous radial motion of the gas sphere under the action of a periodic motion of the spherical piston.

This work was supported by the Russian Foundation for Basic Research (Grant No. 02-01-00550) and the Councils for Supporting the Leading Scientific Schools (Grant No. 00-15-96163).

REFERENCES

1. L. V. Ovsyannikov, *Lectures on Fundamentals of Gas Dynamics* [in Russian], Nauka, Moscow (1981).
2. A. N. Kraiko, "Unrestricted cumulation during one-dimensional unsteady compression of an ideal gas," *Prikl. Mat. Mekh.*, **60**, No. 6, 1000–1007 (1994).
3. L. V. Ovsyannikov, "SUBMODELS program. Gas dynamics," *Prikl. Mat. Mekh.*, **58**, No. 4, 30–55 (1994).
4. L. V. Ovsyannikov, "Summarizing of SUBMODELS application to gas-dynamic equations," *Prikl. Mat. Mekh.*, **63**, No. 3, 362–372 (1999).
5. L. V. Ovsyannikov, "Property of x-autonomy," *Dokl. Ross. Akad. Nauk*, **330**, No. 5, 559–561 (1993).
6. K. V. Brushlinskii and Ya. M. Kazhdan, "Self-similar solutions of some problems of gas dynamics," *Usp. Mat. Nauk*, **18**, No. 2 (110), 3–23 (1963).
7. Ya. M. Kazhdan, "Adiabatic compression of a gas by means of a spherical driver," *J. Appl. Mech. Tech. Phys.*, **18**, No. 1, 18–23 (1977).
8. R. Courant and K. Friedrichs, *Supersonic Flow and Shock Waves*, Interscience, New York (1948).